

Symplectic Convexity for Orbifolds ^{*†}

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Abstract

We generalize symplectic convexity theorems for Hamiltonian actions with proper momentum maps to symplectic actions on orbifolds with mod- Γ proper momentum maps.

§1. Introduction

An orbifold is a Hausdorff topological space locally modelled on \mathbb{R}^n modulo finite group actions. If the group actions are trivial, we recover the concept of manifold. Quite an interesting thing is that the enlarged category is closed under the quotients by finite groups. In symplectic geometry, an important construction of symplectic quotients called Marsden-Weinstein quotients, generically, are not manifolds but symplectic orbifolds [1]. Naturally, we would like to generalize some basic results on symplectic manifolds to the orbifold cases.

Atiyah, independently, Guillemin and Sternberg established symplectic convexity theorems for Hamiltonian torus actions on symplectic manifolds in [2, 3, 4]. Lerman and Tolman got the orbifold versions in [1]. In this note we give some generalizations of their theorems using different methods.

Theorem 1.1 *Let T be a torus and (M, ω) a connected symplectic T -orbifold. Let $\widetilde{M} \longrightarrow M$ be the universal branch covering orbifold and $\Gamma = \pi_1^{\text{orb}}(M)$ the orbifold fundamental group. Assume*

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there is a momentum map $\tilde{J} : \tilde{M} \longrightarrow \mathfrak{t}^*$ for the lifted action. If \tilde{J} is $\text{mod-}\Gamma$ proper and the lifted T -action commutes with that of Γ , then $\tilde{J}(\tilde{M})$ is a closed convex set and $\tilde{J} : \tilde{M} \longrightarrow \tilde{J}(\tilde{M})$ is an open, fibre connected map.

Non-abelian version of Theorem 1.1 in Kähler and projective algebraic manifold cases were independently proved by Guillemin and Sternberg in [4] and Mumford in [5]. Kirwan [6] firstly accomplished the proof for non-abelian Hamilton action of a compact Lie group on an arbitrary connected compact symplectic manifold. Sjamaar [7], Heinzner-Huckleberry [8] discussed extensions in algebraic and Kähler spaces. Flaschka-Ratiu [9] extended the results to the setting of *Poisson actions* of compact Poisson-Lie groups on symplectic manifolds. If it is *symplectic action* on orbifold, we have the following extension:

Theorem 1.2 *Let G be a connected compact Lie group and (M, ω) a connected symplectic G -orbifold. Let $\tilde{M} \longrightarrow M$ be the universal branch covering orbifold and $\Gamma = \pi_1^{\text{orb}}(M)$ the orbifold fundamental group. Assume there is a G -equivariant momentum map $\tilde{J} : \tilde{M} \longrightarrow \mathfrak{g}^*$. If \tilde{J} is $\text{mod-}\Gamma$ proper and the lifted G -action commutes with that of Γ , then $\tilde{J}(\tilde{M}) \cap \mathfrak{t}_+^*$ is a closed convex set and $\tilde{J} : \tilde{M} \longrightarrow \tilde{J}(\tilde{M})$ is a fibre connected map.*

There are several ways to show symplectic convexity theorems. Atiyah, Guillemin and Sternberg, Lerman and Tolman, Kirwan in [2, 3, 4, 1, 6], employed Morse theory. It is easy to show, using a normal form for Hamiltonian action, the momentum map is locally convex [2, 3]. The Morse theory gives rise to a global convexity theorem. Hilgert-Neeb-Plank [10] offered another proof by using a ‘local-global-principle’, dropped compactness of the acted manifold by an assumption that the momentum map is proper. Lerman-Meinrenken-Tolman-Woodward [11], used the symplectic cutting technique. It is a kind of symplectic compactification. Intuitively, by cutting out infinity and collapsing the incision to a point we get a compact symplectic space such that the original non-compact symplectic space is equivariantly embedded in it as an open submanifold. Thus the proof is reduced to the compact case. This method work well in orbifold cases and as a result the symplectic convexity theorems are extended to non-compact orbifold cases [11].

In this paper, we use the techniques developed in [10] by Hilgert-Neeb-Plank where the author dealt with the manifold cases. This proof is more analytical and elementary, it uses least knowledge

of symplectic geometry. In fact, we only need to know that momentum map is locally convex, locally open, locally fiber connected. But these are easily understood if we know the symplectic version of slice theorem for smooth groups actions. Furthermore, this proof tells us clearly what causes the convexity and why it should be so. To some extent, it builds the symplectic convexity theorems on set theoretic topology.

Here is a brief description of the structure of this paper. In Section 2 we review some basic concepts and explain the connections between symplectic action and Hamilton action. Following the same idea of Hilgert-Neeb-Plank, we define a map quotient X_f for mod- Γ proper map in Section 3 and prove that it is a Hausdorff space. Finally we give proofs of Theorem 1.1 and Theorem 1.2 in the last two sections.

§2. Symplectic actions and Hamilton actions

We refer to the Chapter 13 of [15] for a nice account of orbifolds and to [1] for definitions of symplectic orbifolds and Hamiltonian actions on them. Let G be a connected Lie group with Lie algebra \mathfrak{g} and M a smooth connected G -orbifold with symplectic structure ω . A smooth action $G \times M \longrightarrow M$ is called *symplectic* if ω is invariant under the action of G . In this case M is called a *symplectic G -orbifold*. A symplectic action is called *Hamiltonian* action if there exists a map $J : M \longrightarrow \mathfrak{g}^*$, called a *momentum map*, such that

$$i(\xi_M)\omega = dJ_\xi,$$

where ξ_M is the infinitesimal generator corresponding to $\xi \in \mathfrak{g}$ and $J_\xi = \langle J, \xi \rangle$ denotes its ξ -component. A momentum map is called *equivariant* if it intertwines the action of G on M and the coadjoint action of G on \mathfrak{g}^* .

A symplectic action is not always a Hamilton action. However, note that $i(\xi_M)\omega$ is a closed form and $i([\xi, \eta]_M)\omega = -d(\omega(\xi_M, \eta_M))$, so if $H^1(M, \mathbb{R}) = 0$ or G is semi-simple, the symplectic action is a Hamilton action.

Proposition 2.1 *Let (M, ω) be a connected symplectic G -orbifold, and $G = R[G, G]$ a Levi-Malcev decomposition, here R denotes the radical of G . Then*

- (i) *there exists a $[G, G]$ -equivariant momentum map $J : M \longrightarrow \mathfrak{g}^*$;*

(ii) the G -action is Hamiltonian if and only if the R -action is Hamiltonian;

(iii) if M is a compact Kähler manifold with positive Ricci curvature, in particular, if M is Fano, then the G -action is Hamiltonian;

(iv) if M is a compact Kähler manifold, and G acts holomorphically symplectic on M , then the G -action on M is Hamiltonian if and only if the R -action on the Albanese variety $\text{Alb}(M) = H^0(M, \Omega^1)^*/H_1(M, \mathbb{Z})$ is trivial. In particular, if $b_1(M) = 0$, the G -action is Hamiltonian.

Proof. (i) and (ii) followed the discussions above; for (iii), note in this case, $\pi_1(M)$ is a finite group, so we have $H^1(M, \mathbb{R}) = 0$.

For (iv), first suppose that the G -action is Hamiltonian. Recall that the Albanese map $\alpha : M \rightarrow \text{Alb}(M)$ is equivariant. To show the R -action on $\text{Alb}(M)$ is trivial, it suffices to show that every 1-parameter subgroup $r(t) := \langle \exp(t\xi) \rangle \subset R$ has a fix point on M (cf. [12, Proposition 1]) here $\xi \in \mathfrak{r}$ and \mathfrak{r} denotes the Lie algebra of R . Let $J : M \rightarrow \mathfrak{r}^*$ be the momentum map. Then the critical points of function $J_\xi = \langle J, \xi \rangle$, which always exist since M is compact, are the fixed points of $r(t)$. Conversely, Suppose R acts trivially on $\text{Alb}(M)$, then R has fixed points in every fibre of $\alpha^{-1}(\alpha(x))$ (cf. [13, Proposition]). Thus ξ_M has a zero point somewhere on M for any $\xi \in \mathfrak{r}$. Let $\Xi = 1/2(\xi_M - i\mathbb{J}\xi_M)$ be the holomorphic vector field defined by ξ_M , here \mathbb{J} is the complex structure of M . By (iii) of Theorem 1 in [14], there is a function $f \in C^\infty(M, \mathbb{C})$ such that $i(\Xi)\omega = \bar{\partial}f$, so $i(\xi_M)\omega = i(\Xi)\omega + \overline{i(\Xi)\omega} = \bar{\partial}f + \partial\bar{f}$. Let $f = \frac{1}{2}(g + ih)$, where $g, h \in C^\infty(M, \mathbb{R})$. Then $i(\xi_M)\omega = dg$. So the R action on M is Hamiltonian. \square

In the following, let M be an orbifold and $p : \widetilde{M} \rightarrow M$ be the universal branch cover. Then in general case \widetilde{M} is only an orbifold (cf. [15, Chapter 13]). If \widetilde{M} is a manifold then M is called a *good* orbifold. Let $\Gamma := \pi_1^{\text{orb}}(M)$ be the orbifold fundamental group of M . Then Γ is a quotient group of $\pi_1(M_0)$ (cf. [15, Chapter 13]), where M_0 is the regular points of M . The action of G lifted naturally on \widetilde{M} . Let $\tilde{\omega} := p^*\omega$. Then $(\widetilde{M}, \tilde{\omega})$ is also a symplectic G -orbifold. By Proposition 2.1 there always exists a momentum map $\tilde{J} : \widetilde{M} \rightarrow \mathfrak{g}^*$. Still denote \mathfrak{r} the radical of \mathfrak{g} . Then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{r}$, the annihilator of $[\mathfrak{g}, \mathfrak{g}]$ in \mathfrak{g}^* is $[\mathfrak{g}, \mathfrak{g}]^\circ = \mathfrak{r}^*$.

Proposition 2.2 *Suppose that the actions of G and Γ commute. Consider \mathfrak{g}^* as a vector group,*

then there exists a homomorphism $h : \Gamma \longrightarrow \mathfrak{g}^*$ such that

$$\tilde{J}(\gamma \cdot \tilde{x}) - \tilde{J}(\tilde{x}) = h(\gamma), \quad \forall \tilde{x} \in \widetilde{M}.$$

If \tilde{J} is G -equivariant, then $h(\widetilde{M}) \subset \mathfrak{r}^*$. In particular, if G is semisimple, then \tilde{J} factors through \mathfrak{p} so that there exists a G -equivariant momentum map $J : M \longrightarrow \mathfrak{g}^*$ such that $\tilde{J} = J \circ \mathfrak{p}$.

Proof. For any $\gamma \in \Gamma$ and $\xi \in \mathfrak{g}$ set $h_{\xi, \gamma} := \tilde{J}_\xi \circ \gamma - \tilde{J}_\xi$. If we denote \tilde{H}_f the Hamiltonian vector field associated a function f on \widetilde{M} , then

$$\begin{aligned} \tilde{\omega}(\tilde{x})(\tilde{H}_{h_{\xi, \gamma}}(\tilde{x}), \tilde{X}(\tilde{x})) &= \langle d(\tilde{J}_\xi \circ \gamma)(\tilde{x}), \tilde{X}(\tilde{x}) \rangle - \langle d\tilde{J}_\xi(\tilde{x}), \tilde{X}(\tilde{x}) \rangle \\ &= \langle d\tilde{J}_\xi(\gamma\tilde{x}) \circ d\gamma(\tilde{x}), \tilde{X}(\tilde{x}) \rangle - \langle d\tilde{J}_\xi(\tilde{x}), \tilde{X}(\tilde{x}) \rangle \\ &= \langle d\tilde{J}_\xi(\gamma\tilde{x}), d\gamma(\tilde{x})(\tilde{X}(\tilde{x})) \rangle - \langle d\tilde{J}_\xi(\tilde{x}), \tilde{X}(\tilde{x}) \rangle \\ &= \tilde{\omega}(\tilde{x})(\xi_{\tilde{M}}(\gamma\tilde{x}), d\gamma(\tilde{X}(\tilde{x}))) - \tilde{\omega}(\tilde{x})(\xi_{\tilde{M}}(\tilde{x}), \tilde{X}(\tilde{x})) \\ &= (\gamma^*\tilde{\omega})(\tilde{x})(\xi_{\tilde{M}}(\tilde{x}), \tilde{X}(\tilde{x})) - \tilde{\omega}(\tilde{x})(\xi_{\tilde{M}}(\tilde{x}), \tilde{X}(\tilde{x})) \\ &= 0 \end{aligned}$$

since $\mathfrak{p} \circ \gamma = \mathfrak{p}$ implies $\gamma^*\tilde{\omega} = \gamma^*(\mathfrak{p}^*\omega) = (\mathfrak{p} \circ \gamma)^*\omega = \mathfrak{p}^*\omega = \tilde{\omega}$. So $h_{\xi, \gamma}$ is independent of \tilde{x} and we can define $h(\gamma) \in \mathfrak{g}^*$ via $\langle h(\gamma), \xi \rangle = h_{\xi, \gamma}$. Clearly we have $h(\gamma) = \tilde{J} \circ \gamma - \tilde{J}$.

For any $\tilde{x} \in \widetilde{M}$, clearly we have $h(\gamma_1\gamma_2\tilde{x}) = \tilde{J}(\gamma_1\gamma_2\tilde{x}) - \tilde{J}(\tilde{x}) = (\tilde{J}(\gamma_1(\gamma_2\tilde{x})) - \tilde{J}(\gamma_2\tilde{x})) + (\tilde{J}(\gamma_2\tilde{x}) - \tilde{J}(\tilde{x}))$, so $h(\gamma_1\gamma_2) = h(\gamma_1) + h(\gamma_2)$ and h is a homomorphism.

If \tilde{J} is G -equivariant, then for any $\tilde{x} \in \widetilde{M}$, we have $Ad_g^*(h(\gamma)) = Ad_g^*(h(\gamma\tilde{x})) = \tilde{J}(g\gamma\tilde{x}) - \tilde{J}(g\tilde{x}) = \tilde{J}(\gamma g\tilde{x}) - \tilde{J}(g\tilde{x}) = h(\gamma g\tilde{x}) = h(\gamma)$, so $h(\widetilde{M}) \subset \mathfrak{g}^{*G}$. Since $\mu \in \mathfrak{g}^{*G}$ if and only if $ad_\xi^*\mu = 0$ for any $\xi \in \mathfrak{g}$, that is $\langle \mu, [\xi, \eta] \rangle = 0$ for any ξ and $\eta \in \mathfrak{g}$. Hence $h(\Gamma) \subset (\mathfrak{g}^*)^G = [\mathfrak{g}, \mathfrak{g}]^\circ \cong \mathfrak{r}^*$.

If G is semisimple, then $\mathfrak{r} = 0$. Thus $\tilde{J}(\gamma\tilde{x}) = \tilde{J}(\tilde{x})$ for any $\gamma \in \Gamma$. It follows that the G -equivariant momentum map of \widetilde{M} descends to be a G -equivariant map $J : M = \widetilde{M}/\Gamma \longrightarrow \mathfrak{g}^*$.

□

§3. Quotient Space Modulo mod- Γ Map

Let X and Y be topological spaces and $f : X \longrightarrow Y$ a continuous map, f is called *locally fibre connected* (cf. [10, Definition 3.1]) if for any $x \in Y$ there exist a neighborhood U of x such that $f^{-1}(f(x)) \cap U$ is connected for all $x \in U$. If f is locally fibre connected map, the connected component

of the fibre $f^{-1}(f(x))$ passing through x , denoted by F_x , is called a *leaf* of f . Define an equivalence relation \sim on X by saying $x \sim y$ iff they belong to the same leaf. Let X_f denote the quotient space with the quotient topology by shrinking each leaf of f to be a point. Then the quotient map, denoted by π_f , is a continuous map. In general the structure of X_f is very complicate. For example, to assure X_f be a Hausdorff space, the *equivalence relation set* $E =: \{(x, y) \in X \times X | x \sim y\}$ must be a closed subset. If π_f is an open map, X_f is Hausdorff iff E is closed. In [10], it is proved that if Y is a Euclid vector space and f is a proper, locally fibre connected and locally open, then X_f is Hausdorff. In the following we will give a generalization of their result.

Definition 3.1 Let H and L be topological groups, X a locally compact topological H -space and Y a locally compact topological L -space. Let $f : X \longrightarrow Y$ be a continuous map and $\rho : H \longrightarrow L$ be a continuous homomorphism. f is called *mod- H proper* if for any compact subset C of Y , there exist a compact subset B of P such that $f^{-1}(C) \subset H \cdot B$; and f is called *ρ -equivariant*, if $f(g \cdot x) = \rho(g) \cdot f(x)$ for any $g \in H$ and $x \in X$.

Proposition 3.1 Let (X, d_X) and (Y, d_Y) be locally compact metric spaces. Assume that a locally compact group Γ acts isometrically on X and acts on Y via an action homomorphism $\rho : H \longrightarrow \text{Iso}(Y)$. Suppose that $f : X \longrightarrow Y$ is a locally fibre connected, locally open, mod- Γ proper, ρ -equivariant, continuous map. Then X_f is a Hausdorff topological space.

Clearly if Γ is a trivial group we recovered the result of Hilgert-Neeb-Plank. To prove Proposition 3.1, we need the following Lemma 3.1. For this we first give some notions that we will use. For any closed subsets A, B of X , let $d_X(x, B) := \inf_{y \in B} d_X(x, y)$ denote the distance from $x \in X$ to B and $d(A, B) := \sup_{x \in A} d_X(x, B)$. The *Hausdorff distance* between A and B is defined by

$$d^H(A, B) := \max\{d(A, B), d(B, A)\}.$$

Note that $d^H(A, B) = 0$ if and only if $A = B$.

Lemma 3.1 Let C be a compact subset of Y and $W := f^{-1}(C) = \Gamma V$, here V is a compact subset of X . Then

- (i) $\exists \delta > 0, \forall x, y \in W$, if $d_X(x, y) \leq \delta$ and $f(x) = f(y)$, then $F_x = F_y$;
- (ii) $\exists \delta > 0, \forall \epsilon \in [0, \delta], \exists \delta_0 > 0, \forall x, y \in W$, if $d_X(x, y) \leq \delta_0$ and $d_Y(f(x), f(y)) \leq \delta$, then $d_X(x, F_y) \leq \epsilon$;

- (iii) $\forall \eta > 0, \exists \alpha > 0, \forall x' \in V$ and $y' \in W$, if $d_X(x', y') \leq \alpha$, then $d_Y(f(x'), f(y')) \leq \eta$;
- (iv) $\forall \epsilon > 0, \exists \beta > 0, \forall x, y \in W$, if $d_X(x, y) \leq \beta$, then $d(F_x, F_y) \leq \epsilon$.

Proof. Since f is ρ -equivariant, we have,

$$d_Y(f(\gamma \cdot x), f(\gamma \cdot y)) = d_Y(\rho(\gamma) \cdot f(x), \rho(\gamma) \cdot f(y)) = d_Y(f(x), f(y)).$$

Clearly $\gamma \cdot F_x = F_{\gamma \cdot x}$, so $F_x = F_y$ iff $\gamma \cdot F_x = \gamma \cdot F_y$. These facts will be used throughout the following proof.

(i) We prove it by a contradiction. If otherwise, we can find series $\{x_n\}, \{y_n\}$ in W such that $\lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$ and $f(x_n) = f(y_n)$, but $F_{x_n} \cap F_{y_n} = \emptyset$. Note that f is mod- Γ proper, we assume $x_n = \gamma_n x'_n$ such that x'_n vary in a compact subset and set $y_n = \gamma_n y'_n$. Then $f(x'_n) = f(y'_n)$ and $F_{x'_n} \cap F_{y'_n} = \emptyset$.

On the other hand, by extracting out subsequence if necessary, we could assume $\lim_{n \rightarrow \infty} x'_n = x'_\infty \in W$. Since $\lim_{n \rightarrow \infty} d_X(x'_n, y'_n) = \lim_{n \rightarrow \infty} d_X(x_n, y_n) = 0$, we get $\lim_{n \rightarrow \infty} y'_n = x'_\infty$. So $\lim_{n \rightarrow \infty} F_{x'_n} \cap F_{y'_n} = F_{x'_\infty} \neq \emptyset$.

(ii) If otherwise, we can find $\epsilon_0 > 0$ and series $\{x_n\}, \{y_n\}$ such that for any $\delta > 0$ and $d_X(x_n, y_n) \leq \delta$ and $d_Y(f(x_n), f(y_n)) \rightarrow 0$, but $d_X(x_n, F_{y_n}) \geq \epsilon_0$. Similar to the proof of (i), without loss of generality, we may assume $x_n \rightarrow x_\infty$ and $y_n \rightarrow y_\infty$, so $d_X(x_\infty, y_\infty) \leq \delta$ and $f(x_\infty) = f(y_\infty)$. By (i), if δ is small enough then we have $F_{x_\infty} = F_{y_\infty}$. Which means x_∞ and y_∞ lie in the same leaf of f . So $\lim_{n \rightarrow \infty} d_M(x_n, F_{y_\infty}) = 0$. A contradiction.

(iii) Since f is continuous, by definition, $\forall \eta > 0$ and $\forall x' \in V$, there exists $\alpha_{x'} > 0$ such that $\forall y' \in W$ if $d_X(x', y') \leq \alpha_{x'}$ then $d_Y(f(x'), f(y')) \leq \eta$. Let $\Sigma = \bigcup_{x' \in V} \{y' \in W | d_X(x', y') \leq \alpha_{x'}\} \cap V$ be an open cover of V . Note V is compact, by Heine-Borel Covering Theorem there is a finite subcover $\Sigma' = \bigcup_{x'_i \in V} \{y' \in W | d_X(x'_i, y') \leq \alpha_i, i = 1, \dots, n\} \cap V = V$. Set $\alpha = \min\{\alpha_i | i = 1, \dots, n\}$. So for any $x' \in Y$ and $y' \in W$ if $d_Y(x', y') \leq \alpha$, we have $d_Y(f(x'), f(y')) \leq \eta$.

(iv) Let δ and δ_0 as in (ii). By (iii) there exists $\beta > 0$, if $d_X(x, y) \leq \beta$ then $d_Y(f(x), f(y)) \leq \delta$. Now let $\alpha = \min\{\delta_0, \beta\}$ and $\epsilon \in [0, \alpha/2]$, set $E_x := \{z \in F_x | d_X(z, F_y) \leq \epsilon\}$. Then E_x is non-empty closed subset of F_x . If we can show E_x is open in F_x , then (iv) follows.

In fact, let $z \in E_x$ and $w \in F_x \cap \{w \in W | d_X(w, z) \leq \epsilon\}$. Let $w' \in F_y$ such that $d_X(z, w') \leq \epsilon$. Then $d_X(w, w') \leq 2\epsilon \leq \alpha$ and $d_Y(f(w), f(w')) \leq \delta$. By (ii) we have $d_Y(w, F_y) \leq \epsilon$. Thus E_x is open

in F_x . □

Proof of Proposition 3.1. Since f is locally open, π_f is an open map. It suffices to show the equivalence relation set is closed, that is to show for any series $\{x_n\}, \{y_n\}$ in X , if $x_n \rightarrow x_\infty$ and $y_n \rightarrow y_\infty$ and $F_{x_n} = F_{y_n}$, we have $F_{x_\infty} = F_{y_\infty}$. In fact let $C = \{f(x_n), f(y_n) | n = 1, 2, \dots, \infty\}$, then C is a compact subset of Y . Let $W := f^{-1}(C)$. Using (iv) of Lemma 3.1, we have $0 \leq d^H(F_{x_\infty}, F_{y_\infty}) \leq d^H(F_{x_\infty}, F_{x_n}) + d^H(F_{x_n}, F_{y_n}) + d^H(F_{y_n}, F_{y_\infty}) \rightarrow 0$ as $n \rightarrow \infty$. So $F_{x_\infty} = F_{y_\infty}$. Thus X_f is a Hausdorff topological space. □

§4. Abelian Convexity

From now on we will continue the discussions in Section 2. In the following, we use the same notations and terminologies as in Section 3 unless otherwise is especially stressed. Let M be a symplectic orbifold and \widetilde{M} its universal branch covering orbifold and $\Gamma = \pi_1^{\text{orb}}(M)$ the orbifold fundamental group. Assume $G = T$ is a torus, and $T \times M \rightarrow M$ is a symplectic action and the lift $T \times \widetilde{M} \rightarrow \widetilde{M}$ is a Hamilton action with momentum map \tilde{J} . Then \tilde{J} is locally fibre connected, locally open, continuous map. For these properties we refer to [1, 3, 10] for detailed accounts. We suppose that \tilde{J} is mod- Γ proper. Choose a Riemannian metric on M , we may lift it to \widetilde{M} and assume Γ acts isometrically on \widetilde{M} relative to the lifted metric. Let $\pi_{\tilde{J}} : M \rightarrow \widetilde{M}_{\tilde{J}}$ be the quotient map. $\tilde{J} : \widetilde{M} \rightarrow \mathfrak{t}^*$ induces a map $\tilde{J}^a : \widetilde{M}_{\tilde{J}} \rightarrow \mathfrak{t}^*$ such that $\tilde{J}^a \circ \pi_{\tilde{J}} = \tilde{J}$.

Proposition 4.1 *$\widetilde{M}_{\tilde{J}}$ is a Hausdorff topological space.*

Proof. We take \mathfrak{t}^* as a metric space with Euclid metric. By Proposition 2.2, $\tilde{J} \circ \gamma = \tilde{J} + h(\gamma)$. If we consider Γ as an isometry transformation group acting on \mathfrak{t}^* by translation via homomorphism h , then \tilde{J} is a h -equivariant map. Thus we can use Proposition 3.1 to conclude $\widetilde{M}_{\tilde{J}}$ is a Hausdorff topological space. □

Any $\gamma \in \Gamma$ descends to be a homeomorphism of $\widetilde{M}_{\tilde{J}}$, denoted by $\tilde{\gamma}$, satisfying $\tilde{\gamma} \circ \pi_{\tilde{J}} = \pi_{\tilde{J}} \circ \gamma$. Let $\tilde{\Gamma} := \{\tilde{\gamma} | \gamma \in \Gamma\}$. Then $\widetilde{M}_{\tilde{J}}$ is Hausdorff topological space by Proposition 3.1, so $\tilde{J}^a : \widetilde{M}_{\tilde{J}} \rightarrow \mathfrak{t}^*$ is mod- $\tilde{\Gamma}$ proper continuous map. Moreover $\tilde{J}^a \circ \tilde{\gamma} - \tilde{J}^a$ is a constant function on $\widetilde{M}_{\tilde{J}}$ for any $\tilde{\gamma} \in \tilde{\Gamma}$ by Proposition 2.2.

A continuous map $c : [0, 1] \longrightarrow \widetilde{M}_J$ is called a *regular curve* (cf. [10, Definition 3.6]) if $\tilde{J}^q \circ c$ is piecewise differentiable. The length of $\tilde{J}^q \circ c$ is denoted by $l(\tilde{J}^q \circ c)$. For any $\tilde{x}_0^q, \tilde{x}_1^q \in \widetilde{M}_J$, let

$$d(\tilde{x}_0^q, \tilde{x}_1^q) := \inf\{l(\tilde{J}^q \circ c) \mid c \text{ is a regular curve, } c(i) = \tilde{x}_i^q, i = 0, 1\}.$$

Clearly d is symmetric and satisfies the triangle inequality, and $d_{t^*}(\tilde{J}^q(\tilde{x}_0^q), \tilde{J}^q(\tilde{x}_1^q)) \leq d(\tilde{x}_0^q, \tilde{x}_1^q)$.

Proposition 4.2 *For any $\tilde{x}^q \in \widetilde{M}_J$ and $r > 0$, the closed ball $B_r(\tilde{x}^q) := \{\tilde{y}^q \in \widetilde{M}_J \mid d(\tilde{y}^q, \tilde{x}^q) \leq r\}$ is compact.*

Proof. For any $\tilde{x}^q \in \widetilde{M}_J$ and $r_0 > 0$, let $B = B_{r_0}(\tilde{J}^q(\tilde{x}^q))$ be a closed ball in t^* , then there exist a compact subset A of \widetilde{M}_J such that $(\tilde{J}^q)^{-1}(B) \subset \tilde{\Gamma} \cdot A$. Let A_0 denote the connected component of $\tilde{\Gamma} \cdot A$ containing \tilde{x}^q , then A_0 is a compact neighborhood of \tilde{x}^q . So \widetilde{M}_J is locally compact. Furthermore, $\exists \delta > 0$ such that $B_\delta(\tilde{x}^q) \subset A_0$, so $B_\delta(\tilde{x}^q)$ is compact. We can find finite such closed ball $B_\delta(\tilde{x}^q)$ to cover $B_r(\tilde{x}^q)$, hence $B_r(\tilde{x}^q)$ is compact. \square

Proposition 4.3 *$d : \widetilde{M}_J \times \widetilde{M}_J \longrightarrow \mathbb{R}$ is a metric.*

Proof. It suffices to show d separate points. We assume that $d(\tilde{x}^q, \tilde{y}^q) = 0$. Then $\mu := \tilde{J}^q(\tilde{x}^q) = \tilde{J}^q(\tilde{y}^q)$. Let $B := \tilde{J}^{q-1}(\mu)$ and $C := \tilde{J}^{-1}(\mu) = \pi_J^{-1}(B) = \{F_{x_i} \mid F_{x_i} \cap F_{x_j} = \emptyset \text{ if } i \neq j\}$. We claim that $\{x_i\}$ has no convergence point. Otherwise $x_n \rightarrow x_\infty$, by (i) of Proposition 3.1, $\exists N > 0$ such that when $n > N$ all F_{x_n} coincide, a contradiction. Thus we can find disjoint open sets $\{U_i^q\}$ in \widetilde{M}_J , such that each U_i^q contains only one element \tilde{x}_i^q of $\tilde{J}^{q-1}(\mu)$. Since \tilde{J} is a locally convex map, clearly so is \tilde{J}^q . So $\tilde{J}^q(U_i^q)$ contain a closed ball with positive radius ϵ and any regular curve c starts at x_i^q and leaves U_i^q satisfies $l(\tilde{J}^q \circ c) \geq \epsilon$. So \tilde{x}^q, \tilde{y}^q must lie in the same closed ball. This shows $\tilde{x}^q = \tilde{y}^q$. \square

Remark 4.1 Proposition 4.2 and 4.3 together say \widetilde{M}_J is a connected locally compact metric space. Note that a metric space is not necessary locally compact. The simplest example is the rational number \mathbb{Q} as a subspace of \mathbb{R}^1 with Euclid metric, clearly it is not locally compact since any compact subset of \mathbb{Q} is a finite set. A connected Hausdorff space is not necessary locally compact, too. For example, the quotient of \mathbb{R}^1 modulo the equivalence relation set $E = \mathbb{Z} \times \mathbb{Z}$ is clearly connected but not locally compact.

Proof of Theorem 1.1. Fix $\tilde{x}^q, \tilde{y}^q \in \widetilde{M}_J$ and let $d_0 := d(\tilde{x}^q, \tilde{y}^q)$. For any $n \in \mathbb{N}$, there exist a regular

curve c_n connecting \tilde{x}_0^q and \tilde{x}_1^q with $l(\tilde{J}^q \circ c_n) \leq d_0 + \frac{1}{n}$. Let $\tilde{x}_{1/2}^{q;n}$ be the midpoints of c_n . They are contained in the ball $B_{2d_0}(\tilde{x}^q)$ which is compact, so they have a coherent point $\tilde{x}_{1/2}^q$. This point satisfies

$$d(\tilde{x}^q, \tilde{x}_{1/2}^q) = d(\tilde{x}_{1/2}^q, \tilde{y}^q) = d_0/2.$$

Repeat this process for the pairs of points $(\tilde{x}^q, \tilde{x}_{1/2}^q)$ and $(\tilde{x}_{1/2}^q, \tilde{y}^q)$ to obtain $\tilde{x}_{1/4}^q$ and $\tilde{x}_{3/2}^q$ respectively, satisfying

$$d(\tilde{x}^q, \tilde{x}_{1/4}^q) = d(\tilde{x}_{1/4}^q, \tilde{x}_{1/2}^q) = d(\tilde{x}_{1/2}^q, \tilde{x}_{3/4}^q) = d(\tilde{x}_{3/4}^q, \tilde{y}^q).$$

Inductively we find points $\tilde{x}_{k/2^m}^q$, for $0 \leq k \leq 2^m$ such that

$$d(\tilde{x}_{k/2^m}^q, \tilde{x}_{k'/2^m}^q) = d_0 |k/2^m - k'/2^m|.$$

So we can extend $k/2^m \mapsto \tilde{x}_{k/2^m}^q$ to a continuous map $c : [0, 1] \rightarrow \widetilde{M}_{\tilde{J}}$ such that $d(c(t), c(t')) = d_0 |t - t'|$. This means

$$d_{\mathfrak{t}^*}(\tilde{J}^q \circ c(t), \tilde{J}^q \circ c(t')) = d_0 |t - t'|$$

which can only happen iff $\tilde{J}^q \circ c$ is a straight line. So $\tilde{J}(\widetilde{M}) = \tilde{J}^q(\widetilde{M}_{\tilde{J}})$ is a convex set.

To show that the fibres of \tilde{J} are connected, it suffices to show \tilde{J}^q is injective. We assume $\tilde{J}^q(\tilde{x}^q) = \tilde{J}^q(\tilde{y}^q)$. We construct a regular curve c connecting \tilde{x}^q and \tilde{y}^q as in the previous paragraph. Then $d(\tilde{x}^q, \tilde{y}^q) = d_{\mathfrak{t}^*}(\tilde{J}^q \circ c(0), \tilde{J}^q \circ c(1)) = 0$. So that $\tilde{x}^q = \tilde{y}^q$. In view of what we have already shown, \tilde{J}^q is a homeomorphism. Since $\pi_{\tilde{J}}$ is an open map, $\tilde{J} = \tilde{J}^q \circ \pi_{\tilde{J}}$ is an open map as well. Thus $\tilde{J} : \widetilde{M} \rightarrow \tilde{J}(\widetilde{M})$ is an open, fibre connected map.

For any $\mu \in \overline{\tilde{J}^q(\widetilde{M})}$, assume $\lim_{n \rightarrow +\infty} \tilde{J}^q(\tilde{x}_n^q) = \mu$. Since \tilde{J}^q is a mod- $\tilde{\Gamma}$ proper map, we can assume $\tilde{x}_n^q = \tilde{\gamma} \cdot \tilde{y}_n^q$ such that \tilde{y}_n^q varies in a compact subset. By extracting out subsequence if necessary, we could assume $\lim_{n \rightarrow \infty} \tilde{y}_n^q = \tilde{y}_\infty^q \in \widetilde{M}$. Thus $\mu = \tilde{J}^q(\tilde{y}_\infty^q) + \lim_{n \rightarrow \infty} h(\tilde{\gamma}_n) \in \tilde{J}^q(\widetilde{M}) + h(\tilde{\Gamma}) = \tilde{J}^q(\widetilde{M})$. Hence $\tilde{J}(\widetilde{M})$ is a closed convex set. \square

§5. Non-Abelian Convexity

First we review the symplectic cross-section theorem for actions of compact connected Lie group G . (cf. [3, Theorem 6.4] and [11, Theorem 3.1]). Let $T \subset G$ be a maximal torus and $\mathfrak{t}_+^* \subset \mathfrak{t}^*$ a

positive Weyl chamber. For any $\lambda \in \mathfrak{g}^*$, there is a unique point in \mathfrak{t}_+^* which is the intersection point of the coadjoint orbit $Ad_G^* \lambda$ and \mathfrak{t}_+^* . Thus \mathfrak{t}_+^* parameterizes the coadjoint orbits and is a *section* for the coadjoint action. Now let M be a connected Hamilton G -orbifold with equivariant moment map $J : M \longrightarrow \mathfrak{g}^*$, we can “pull back” the section for coadjoint action to a section for the G -action on M via J . Let σ denote the interior of \mathfrak{t}_+^* . The preimage $Y := J^{-1}(\sigma)$ is a connected T -invariant *suborbifold* of M . The *Symplectic Section Theorem* claim that Y is a *symplectic* suborbifold, thus a “symplectic section”(cf. [11, Theorem 3.1]). It is easy to see the restriction J_Y of J to Y is a moment map for action of T and $G \cdot Y$ is dense in M . Thus the “symplectic section” set up a bridge between torus action and non-abelian group action. We will use these facts to prove Theorem 1.2 via using Theorem 1.1.

Proof of Theorem 1.2 Let σ be the interior of the Weyl chamber \mathfrak{t}_+^* and $\tilde{Y} =: \tilde{J}^{-1}(\sigma)$ the symplectic section. \tilde{Y} is a symplectic T -orbifold with momentum map $\tilde{J}_{\tilde{Y}}$. Since σ is a relative open subset of \mathfrak{t}_+^* , we can choose an ascending sequence of closed subsets $\sigma_i \subset \sigma$ such that $\cup_{i \in \mathbb{N}} \sigma_i = \sigma$. Let $\tilde{Y}_i := \tilde{J}_{\tilde{Y}}^{-1}(\sigma_i)$ be the closed subsets of $\tilde{M}_{\tilde{J}}$. Since \tilde{Y} is connected, we can choose an ascending sequence of connected components \tilde{Y}'_i of \tilde{Y}_i such that $\tilde{Y} = \cup_{i \in \mathbb{N}} \tilde{Y}'_i$. The restriction $J|_{\tilde{Y}'_i} : \tilde{Y}'_i \longrightarrow \sigma_i$ is a mod- $\tilde{\Gamma}$ -proper. Clearly $J|_{\tilde{Y}'_i}$ is also a locally fibre connected, locally convex, locally open map. By Theorem 1.1, we know $\tilde{J}(\tilde{Y}'_i)$ form an ascending sequence of closed convex subsets of σ . Hence $\tilde{J}(\tilde{Y})$ is convex. So $\overline{\tilde{J}(\tilde{M})} \cap \mathfrak{t}_+^* = \overline{\tilde{J}(Y)}$ is convex locally polyhedral set. If we can prove $\tilde{J}(\tilde{M})$ is closed, then $\tilde{J}(\tilde{M}) \cap \mathfrak{t}_+^* = \overline{\tilde{J}(Y)}$ is a closed convex set. In fact, since \tilde{J} is mod- Γ proper, the quotient map $\hat{J} : M = \tilde{M}/\Gamma \longrightarrow \mathfrak{g}^*/h(\Gamma)$ is proper, $\hat{J}(M)$ is closed in $\mathfrak{g}^*/h(\Gamma)$, so $\tilde{J}(\tilde{M})$ is a closed subset of \mathfrak{g}^* .

It remains to show that the fibre $\tilde{J}^{-1}(\mu)$ is connected for any $\mu \in \mathfrak{g}^*$. By Theorem 1.1, the fibres of $\tilde{J}_{\tilde{Y}}$ are connected. Since $G \cdot Y$ is dense in M and \tilde{J} is equivariant, $\tilde{J}|_{G \cdot Y}$ is fibre connected. Clearly $\tilde{J}^{-1}(Ad_G^* \mu) = G \cdot \tilde{J}^{-1}(\mu)$. Since G and G_μ are connected, $\tilde{J}^{-1}(\mu)$ is connected if and only if $\tilde{J}^{-1}(Ad_G^* \mu)$ is connected. So we may assume $\mu \in \mathfrak{t}_+^*$.

Now for any $\mu \in \sigma$, the fibre $\tilde{J}^{-1}(\mu)$ is connected by Theorem 1.1. So for any convex open neighborhood B of μ , the set $G \cdot (\tilde{J}^{-1}(B) \cap \sigma)$ is connected. Since $\tilde{J}^{-1}(Ad_G^* \cdot B \cap \mathfrak{t}_+^*) \cap G \cdot \tilde{J}^{-1}(\sigma) = G \cdot \tilde{J}^{-1}(B \cap \sigma)$, we know $\overline{\tilde{J}^{-1}(Ad_G^* \cdot B \cap \mathfrak{t}_+^*)} = \overline{G \cdot \tilde{J}^{-1}(B \cap \sigma)}$ is connected.

For any $\mu \in \mathfrak{t}_+^*$, let B_i be convex open Neighborhoods with $\mu \in \overline{B_i}$ and $B_{i+1} \subset B_i$ such that $\cap_{i \in \mathbb{N}} \overline{B_i} = \{\mu\}$. Then $\tilde{J}^{-1}(Ad_G^* \mu) = \overline{\cap_{i \in \mathbb{N}} \tilde{J}^{-1}(Ad_G^* \cdot B_i \cap \mathfrak{t}_+^*)}$ is connected. It follows that $\tilde{J}^{-1}(\mu)$ is connected. So \tilde{J} is a fibre connected map. \square

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